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Source: *Technometrics*, Vol. 23, No. 1 (Feb., 1981), pp. 45-57
Published by: American Statistical Association and American Society for Quality
Stable URL: http://www.jstor.org/stable/1267975
Accessed: 03/10/2011 15:21

TECHNOMETRICS ©, VOL. 23, NO. 1, FEBRUARY 1981

Incomplete Block Designs for Comparing Treatments With a Control: General Theory

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In this paper we develop a theory of optimal incomplete block designs for comparing several treatments with a control. This class of designs is appropriate for comparing simultaneously $p \geq 2$ test treatments with a control treatment (the so-called multiple comparisons with a control (MCC) problem) when the observations are taken in incomplete blocks of common size $k < p + 1$. For this problem we propose a new general class of incomplete block designs that are *balanced* with respect to (wrt) *test* treatments. We shall use the abbreviation BTIB to refer to such designs. We study their structure and give some methods of construction. A procedure for making exact joint confidence statements for this multiple comparisons problem is described. By using a new concept of admissibility of designs, it is shown how "inferior" designs can be eliminated from consideration, and attention limited to a small class of BTIB designs that can be constructed from so-called generator designs in the minimal complete class of such designs. Some open problems concerning construction of BTIB designs are posed.

KEY WORDS: Multiple comparisons with a control; Balanced treatment incomplete block (BTIB) designs; Generator designs; Admissible designs; Optimal designs.

1. INTRODUCTION

In many industrial, agricultural, and biological experiments, it is often desired to compare simultaneously several test treatments with a control treatment. The earliest correct work on this problem was carried out by Dunnett (1955, 1964). Dunnett (1955) also posed (but did not solve) the problem of optimally allocating experimental units to control and test treatments so as to maximize the probability associated with the joint confidence statement concerning the many-to-one comparisons between the mean of the control treatment and the means of the test treatments. This optimal allocation problem was solved by Bechhofer and his coworkers (1969, 1970, 1971).

In all of the aforementioned papers it was tacitly assumed that a completely randomized (CR) design was used. However, many practical situations may require the blocking of experimental units in order to cut down on bias and improve the precision of the experiment. If the block size is large enough to accommodate one replication of all of the test treatments and additional control treatments as well, then the design and analysis of replications of the experiment can be carried out using the optimal allocations described in Bechhofer (1969) and Bechhofer and Nocturne (1970) with only the usual modifications.

We shall study the multiple comparisons problem in the situation that commonly occurs in practice, that is, when all of the blocks have a common size but the block size is less than the total number of treatments. Robson (1961) pointed out that Dunnett's procedure can be extended to the case in which a balanced incomplete block (BIB) design between *all* of the treatments (including the *control* treatment) is used. Cox (1958, p. 238) noted that BIB designs are perhaps not appropriate for the multiple comparisons with the control (MCC) problem because of the special role played by the control treatment. He suggested a design that employs the control treatment an equal number of times (once, twice, etc.) in each block, the test treatments forming a BIB design in the remaining plots of the blocks; no analytical details were given for this proposed design. Pešek (1974) has given analytical details for a special case of Cox's design (i.e., the control treatment is employed once in each block); he shows that this design is more efficient than a BIB design for comparisons with a control, but it is less efficient for pairwise comparisons between the test treatments. It should be noted that even Cox's more general design is quite restrictive.

In this paper we propose a new general class of incomplete block designs that is appropriate for the MCC problem. We give the basic theory underlying these designs, and include as well some methods of construction. The method of analysis is described. Admissibility and optimality considerations are discussed in some detail.

2. PRELIMINARIES

Let the treatments be indexed by $0, 1, \ldots, p$ with 0 denoting the control treatment and $1, 2, \ldots, p$ denoting the $p \geq 2$ test treatments. Let $k < p + 1$ denote the common size of each block, and let b denote the number of blocks available for experimentation. Thus $N = kb$ is the total number of experimental units. If treatment i is assigned to the hth plot of the jth block $(0 \leq i \leq p,\ 1 \leq h \leq k,\ 1 \leq j \leq b)$, let Y_{ijh} denote the corresponding random variable; we assume the usual additive linear model (no treatment × block interaction)

$$Y_{ijh} = \mu + \alpha_i + \beta_j + e_{ijh} \tag{2.1}$$

with $\sum_{i=0}^{p} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$; the e_{ijh} are assumed to be iid $N(0, \sigma^2)$ random variables. It is desired to make an exact joint confidence statement (employing one-sided or two-sided intervals) concerning the p differences $\alpha_0 - \alpha_i$ based on their BLUE's $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leq i \leq p)$.

3. CHOICE OF THE CLASS OF DESIGNS

3.1 BTIB Designs

Since it is desired to make a confidence statement that applies simultaneously to all of the p differences $\alpha_0 - \alpha_i$ $(1 \leq i \leq p)$, we shall regard our problem as being symmetric in these differences. To this end, we consider a class of designs for which $\text{var}\{\hat{\alpha}_0 - \hat{\alpha}_i\} = \tau^2\sigma^2$ $(1 \leq i \leq p)$ and $\text{corr}\{\hat{\alpha}_0 - \hat{\alpha}_{i_1}, \hat{\alpha}_0 - \hat{\alpha}_{i_2}\} = \rho$ $(i_1 \neq i_2;\ 1 \leq i_1, i_2 \leq p)$; the parameters τ and ρ depend on the design employed. We shall refer to such designs as *BTIB* designs since they are *balanced* with respect to the *test* treatments. The following theorem states the necessary and sufficient conditions that a design must satisfy in order to be a BTIB design. The proof of this theorem is given in the Appendix; in the process we also derive expressions for $\text{var}\{\hat{\alpha}_0 - \hat{\alpha}_i\}$ and ρ. These quantities play a crucial role in our later considerations.

Theorem 3.1: For given (p, k, b) consider a design with the incidence matrix $\{r_{ij}\}$ where r_{ij} is the number of replications of the ith treatment in the jth block. Let $\lambda_{i_1 i_2} = \sum_{j=1}^{b} r_{i_1 j} r_{i_2 j}$ denote the total number of times that the i_1th treatment appears with the i_2th treatment in the same block over the whole design $(i_1 \neq i_2;\ 0 \leq i_1,\ i_2 \leq p)$. Then the necessary and sufficient conditions for a design to be BTIB are

$$\begin{gathered}\lambda_{01} = \lambda_{02} = \cdots = \lambda_{0p} = \lambda_0 \text{ (say)} \\ \text{and} \\ \lambda_{12} = \lambda_{13} = \cdots = \lambda_{p-1,p} = \lambda_1 \text{ (say)}.\end{gathered} \tag{3.1}$$

In other words, each test treatment must appear with (i.e., in the same block as) the control treatment the same total number of times (λ_0) over the design, and each test treatment must appear with every other test treatment the same total number of times (λ_1) over the design.

Examples of some selected BTIB designs are given in (3.2) through (3.9) and in Section 5. Expressions for $\text{var}\{\hat{\alpha}_0 - \hat{\alpha}_i\}$ and $\text{corr}\{\hat{\alpha}_0 - \hat{\alpha}_{i_1}, \hat{\alpha}_0 - \hat{\alpha}_{i_2}\}$ $(i_1 \neq i_2)$ are given in terms of λ_0 and λ_1 by (4.2) and (4.4), respectively.

Remark 3.1: We note that Theorem 3.1 places no restriction on $r_i = \sum_{j=1}^{b} r_{ij}$ $(1 \leq i \leq p)$, the number of replications of the ith test treatment, and hence a design can be BTIB without the r_i $(1 \leq i \leq p)$ being equal. Such a design for which $(p, k, b) = (4, 3, 7)$ and $\lambda_0 = 2$, $\lambda_1 = 2$ with $r_1 = r_2 = r_3 = 4$, $r_4 = 5$ is given by

$$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & 4 \\ 3 & 4 & 3 & 4 & 3 & 4 & 4 \end{matrix}\right\}. \tag{3.2}$$

3.2 Construction of BTIB Designs

At this point, we indicate several methods of constructing BTIB designs. For a starting point, we introduce the concept of a generator design.

Definition 3.1: For given (p, k) a generator design is a BTIB design no proper subset of whose blocks forms a BTIB design, and no block of which contains only one of the $p + 1$ treatments.

Thus

$$D_0 = \left\{\begin{matrix} 0 & 0 \\ 1 & 2 \end{matrix}\right\}, \quad D_1 = \left\{\begin{matrix} 1 \\ 2 \end{matrix}\right\}, \quad D = \left\{\begin{matrix} 0 & 0 & 1 \\ 1 & 2 & 2 \end{matrix}\right\} \tag{3.3}$$

are BTIB designs with $(\lambda_0, \lambda_1) = (1, 0), (0, 1), (1, 1)$, respectively; however, only D_0 and D_1 are *generator* designs (as is the design given by (3.2)). Design D of (3.3) suggests the role of generator designs. For given (p, k) there are several (often many) generator designs; for example, for each $p \geq 2$, $k = 2$ there are exactly *two* generator designs. (See (3.4).) By taking unions of replications of these generator designs, *at least one of which* has $\lambda_0 > 0$, we obtain an *implementable* BTIB design. The problem of determining how many generator designs exist for arbitrary given (p, k) is an open problem (it can be shown that there are a finite number), which appears to be very formidable; we comment on this problem in Section 6. In the sequel we consider only implementable BTIB designs.

Suppose that for given (p, k) there are n generator designs D_i $(1 \le i \le n)$. Let $\lambda_0^{(i)}$, $\lambda_1^{(i)}$ be the design parameters associated with D_i, and let b_i be the number of blocks required by D_i $(1 \le i \le n)$. Then a BTIB design $D = \bigcup_{i=1}^n f_i D_i$ obtained by forming unions of $f_i \ge 0$ replications of D_i has the design parameters $\lambda_0 = \sum_{i=1}^n f_i \lambda_0^{(i)}$, $\lambda_1 = \sum_{i=1}^n f_i \lambda_1^{(i)}$ and requires $b = \sum_{i=1}^n f_i b_i$ blocks. The set of D_i with $f_i > 0$ will be referred to as the *support* of D.

The following is another example of generator designs (generalizing (3.3)). As noted before, for each $p \ge 2$, $k = 2$ there are exactly two generator designs. They are

$$D_0 = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ 1 & 2 & \cdots & p \end{vmatrix}, \quad D_1 = \begin{vmatrix} 1 & 1 & \cdots & p-1 \\ 2 & 3 & \cdots & p \end{vmatrix}. \tag{3.4}$$

From these generator designs, implementable BTIB designs of the type $D = f_0 D_0 \cup f_1 D_1$ can be constructed for $f_0 \ge 1$, $f_1 \ge 0$; the corresponding design parameters for D are $\lambda_0 = f_0$, $\lambda_1 = f_1$, $b = f_0 p + f_1 p(p-1)/2$.

We now consider several methods of constructing generator designs for $k \ge 3$; this list is not exhaustive.

Method I: The preceding example suggests the following method of constructing a class of generator designs: For given (p, k), a generator design D_m will have $m + 1$ plots in each block assigned to the control treatment; the p test treatments are assigned to the remaining $k - m - 1$ plots of the b_m blocks $(0 \le m \le k - 2)$ in such a way as to form a BIB design. The generator design D_{k-1} contains no control treatments; it consists of a RB (BIB) design between the p test treatments if $p = k (p > k)$. Thus for $(p, k) = (3, 3)$ we have the following three generator designs in this class.

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{Bmatrix},$$

$$D_2 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}. \tag{3.5}$$

Method II: Starting with a BIB design between $t > p$ treatments in b blocks, one can relabel the treatments $p+1, p+2, \ldots, t$ zeros to obtain a new BTIB design with possibly an additional block or blocks, each one of the latter containing only one test treatment or only the control treatment. After deleting all of these one-treatment blocks, and identifying the support of the resulting BTIB design, one obtains the desired generator design(s). For example, we consider the following BIB design for $(t, k, b) = (7, 3, 7)$:

$$\begin{Bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \end{Bmatrix} \tag{3.6}$$

By replacing the sevens by zeros, one obtains a generator design for $(p, k, b) = (6, 3, 7)$ with $\lambda_0 = 1$, $\lambda_1 = 1$. This process can be continued by then replacing the sixes by zeros to obtain a generator design for $(p, k, b) = (5, 3, 7)$ with $\lambda_0 = 2$, $\lambda_1 = 1$; continuing, one can then replace the fives by zeros to obtain a generator design for $(p, k, b) = (4, 3, 7)$ with $\lambda_0 = 3$, $\lambda_1 = 1$. Finally, replacing the fours by zeros, one obtains the union of the two generator designs D_0 and D_1 of (3.5) with a block containing all zeros. The BTIB designs obtained in this way for $k = 3$, $b = 7$ are

$p = 6$, $b = 7$ $(\lambda_0 = 1, \lambda_1 = 1)$:

$$\begin{Bmatrix} 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 2 & 5 & 3 & 4 \\ 3 & 6 & 5 & 4 & 6 & 5 & 6 \end{Bmatrix} \tag{3.7a}$$

$p = 5$, $b = 7$ $(\lambda_0 = 2, \lambda_1 = 1)$:

$$\begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 3 & 4 & 0 & 2 & 3 \\ 3 & 5 & 4 & 5 & 2 & 4 & 5 \end{Bmatrix} \tag{3.7b}$$

$p = 4$, $b = 7$ $(\lambda_0 = 3, \lambda_1 = 1)$:

$$\begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 2 \\ 3 & 3 & 4 & 1 & 2 & 4 & 4 \end{Bmatrix}. \tag{3.7c}$$

Remark 3.2: From (3.7a) it is clear that every BIB design involving t treatments yields a BTIB design with $p = t - 1$ *test* treatments.

Remark 3.3: It is well known (e.g., John 1964) that unequal replicate designs having the same properties as BIB designs can be constructed. Such designs are therefore BTIB designs. An example of such a generator design for $(p, k, b) = (5, 3, 8)$ with $\lambda_0 = 0$, $\lambda_1 = 2$ is given by

$$\begin{Bmatrix} 1 & 1 & 1 & 2 & 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 & 5 & 5 & 5 & 5 \\ 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 \end{Bmatrix}. \tag{3.8}$$

Method III: Consider a group-divisible partially balanced incomplete block (GD-PBIB) design with two associate classes between t treatments in blocks of size k. The association scheme of such a GD-PBIB design can be represented in the form of an $m \times n$ array (with $mn = t$). Any two treatments in the same row of the array are first associates, and those in different rows are second associates. Suppose that $m \geq k$; one can then take $p = m$ and relabel the entries in $n_1 > 0$ columns of the array by 1, 2, ..., p and the entries in the remaining $n_2 = n - n_1 > 0$ columns by zeros, thus obtaining a BTIB design. As with Method II, such a design may not be a generator design and may contain some blocks that must be deleted. After deleting such blocks, a BTIB design is obtained. By identifying the support of this resulting design, the desired generator design(s) are obtained; some (or all) of these can usually be obtained by the previous two methods. If $n \geq k$, one can take $p = n$ and then relabel the entries in $m_1 > 0$ rows of the array by 1, 2, ..., p and the entries in $m_2 = m - m_1 > 0$ rows by zeros, thus obtaining a BTIB design, possibly with blocks that must be deleted.

To see the use of this method, consider the GD-PBIB design #R20 (for $k = 3$, $t = 12$, $m = 4$, $n = 3$, $b = 36$) in the monograph by Bose, Clatworthy, and Shrikhande (1954), which has the following association scheme:

$$\begin{Bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{Bmatrix}$$

By relabeling the treatments 5 through 8 by 1 through 4, and 9 through 12 by zeros, one obtains the union of a BTIB design with a design containing one block with only zeros. After that block has been deleted, the support of the remaining BTIB design consists of the following:

$$\text{2 replications of} \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 \end{Bmatrix}, \quad (3.9a)$$

$$\text{2 replications of} \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{Bmatrix}, \quad (3.9b)$$

$$\text{2 replications of} \begin{Bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{Bmatrix}, \quad (3.9c)$$

$$\text{1 replication of} \begin{Bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 4 \end{Bmatrix}. \quad (3.9d)$$

Thus the designs given by (3.9a) through (3.9d) are generator designs for $p = 4$, $k = 3$; the first three designs are obtainable by Method I, while the fourth is not. Using the definition of inadmissibility of a design as given in Section 5, we shall see that the design of (3.7c) obtained for $(p, k, b) = (4, 3, 7)$ by Method II is inadmissible wrt the design of (3.9d), which has $\lambda_0 = 2$, $\lambda_1 = 2$.

Method IV: Suppose that for given (p, k) we have a generator design D_1 with $\lambda_0 > 0$. Then a new generator design D_2 for the same (p, k) can be obtained by taking a "complement" of D_1 in the following way: Separate the blocks of D_1 in different sets so that each block in a given set has zero assigned in an equal number of plots (0 times, 1 time, etc.). For example, consider the design (3.7c) the blocks of which can be separated into three sets as follows:

$$D_1 = \begin{Bmatrix} 1 & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 2 & \vdots & 1 & 2 & 3 & \vdots & 0 & 0 & 0 \\ 4 & \vdots & 3 & 3 & 4 & \vdots & 1 & 2 & 4 \end{Bmatrix}.$$

For each set of D_1 write its "complementary" set (with zero assigned in the *same* number of plots) so that the union of that set with its complementary set forms a generator design; if $r_{ij} = 0$ or 1 $(1 \leq i \leq p)$ then that union is simply a generator design that can be constructed by Method I. These complementary sets in the present example are

$$\begin{Bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 4 & 4 \end{Bmatrix}, \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 4 & 4 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix};$$

by taking their union we obtain the generator design D_2 given by (3.9d). The b-values for D_1 and its complement D_2 are not in general equal, although in the present example they are.

4. JOINT CONFIDENCE STATEMENTS

4.1 Expressions for Estimates

We first give the expressions for the BLUE $\hat{\alpha}_0 - \hat{\alpha}_i$ of $\alpha_0 - \alpha_i$ $(1 \leq i \leq p)$. Let T_i denote the sum of all observations obtained with the ith treatment $(0 \leq i \leq p)$, and let B_j denote the sum of all observations in the jth block $(1 \leq j \leq b)$. Define $B_i^* = \sum_{j=1}^{b} r_{ij} B_j$ and let $Q_i = kT_i - B_i^*$ $(0 \leq i \leq p)$. Then

$$\hat{\alpha}_0 - \hat{\alpha}_i = \frac{\lambda_1 Q_0 - \lambda_0 Q_i}{\lambda_0(\lambda_0 + p\lambda_1)} \quad (1 \leq i \leq p). \quad (4.1)$$

Also,

$$\text{var}\{\hat{\alpha}_0 - \hat{\alpha}_i\} = \tau^2 \sigma^2 \quad (1 \leq i \leq p) \quad (4.2)$$

where

$$\tau^2 = \frac{k(\lambda_0 + \lambda_1)}{\lambda_0(\lambda_0 + p\lambda_1)}, \quad (4.3)$$

and

$$\rho = \text{corr}\{\hat{\alpha}_0 - \hat{\alpha}_{i_1}, \hat{\alpha}_0 - \hat{\alpha}_{i_2}\} = \frac{\lambda_1}{\lambda_0 + \lambda_1},$$

$$(i_1 \neq i_2;\ 1 \leq i_1, i_2 \leq p). \quad (4.4)$$

Table 1. ***Analysis of Variance Table for BTIB Designs***

Source of Variation	Sum of Squares	d.f.
Treatments (adjusted)	$\frac{Q_0^2\{(p-1)(\lambda_0^2+\lambda_1^2)+(p^2+3)\lambda_0\lambda_1\}}{k(p+1)^2\lambda_0^2(\lambda_0+p\lambda_1)} + \frac{\sum_{i=1}^{p} Q_i^2}{k(\lambda_0+p\lambda_1)}$	p
Blocks	$\frac{1}{k}\sum_{j=1}^{b} B_j^2 - \frac{G^2}{N}$	b-1
Error	(by subtraction)	N-p-b
Total	$H - \frac{G^2}{N}$	N-1

The expressions for (4.1) through (4.4) are derived in the Appendix.

An unbiased estimate s_v^2 of σ^2 based on $v = N - p - b$ degrees of freedom (df) can be computed as $SS_{\text{error}}/(N - p - b)$ where SS_{error} can be obtained by subtraction (as in BIB designs) from Table 1. The expressions in the table are derived in the Appendix; G and H denote the sum and sum of squares, respectively, of all observations. We note that if $\lambda_0 = \lambda_1 = \lambda$ (say), then $SS_{\text{treatments (adjusted)}}$ reduces to $\sum_{i=0}^{p} Q_i^2/k\lambda(p+1)$ (i.e., the *same* expression as for a BIB design); this latter expression thus holds for any *completely balanced* design (such as (3.2), which is *not* a BIB design).

Remark 4.1: For many BTIB designs we have $r_{ij} > 1$, and thus within-block replication occurs. For such designs the sum of squares (SS) for error can be partitioned into SS due to "pure error" and SS due to "interaction," and this decomposition can be used in testing the additivity assumption or the assumption of block-to-block variance homogeneity. Such tests are not pursued in this paper.

Remark 4.2: It is easily shown for BTIB designs that the $\hat{\alpha}_i$ $(1 \le i \le p)$ have a constant variance $= \eta^2\sigma^2$ (say), a common correlation $= \gamma$ (say), and we have the relationship

$$\begin{aligned} \operatorname{var}\{\hat{\alpha}_i - \hat{\alpha}_j\} &= 2(1-\gamma)\eta^2\sigma^2 \\ &= 2(1-\rho)\tau^2\sigma^2 \ (i \ne j;\ 1 \le i, j \le p). \quad (4.5) \end{aligned}$$

Thus the relative precision of the estimators $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \le i \le p)$ for the MCC problem wrt the estimators $\hat{\alpha}_i - \hat{\alpha}_j$ $(i \ne j,\ 1 \le i, j \le p)$ for the pairwise comparisons among the p test treatments is given by

$$\frac{\operatorname{var}\{\hat{\alpha}_i - \hat{\alpha}_j\}}{\operatorname{var}(\hat{\alpha}_0 - \hat{\alpha}_i\}} = 2(1-\rho). \quad (4.6)$$

Note that the relative precision is $\gtrless 1$ depending on whether $\rho \lessgtr 1/2$.

4.2 Confidence Statements

Joint $100(1-\alpha)$ percent confidence intervals for the $\alpha_0 - \alpha_i$ $(1 \le i \le p)$ and for the $\alpha_i - \alpha_j$ $(i \ne j, 1 \le i, j \le p)$ are given below.

I. One-sided confidence intervals. When σ^2 is unknown, the joint one-sided confidence intervals are given by

$$\alpha_0 - \alpha_i \ge \hat{\alpha}_0 - \hat{\alpha}_i - t_{v,p,\rho}^{(\alpha)}\tau s_v \quad (1 \le i \le p). \quad (4.7)$$

In (4.7) $t_{v,p,\rho}^{(\alpha)}$ denotes the upper equicoordinate α point of the p-variate equicorrelated central t distribution with common correlation ρ, and with df v (as defined by Dunnett and Sobel 1954); for tables of $t_{v,p,\rho}^{(\alpha)}$ see Krishnaiah and Armitage (1966).

When σ^2 is known, the joint one-sided confidence intervals are obtained by replacing $t_{v,p,\rho}^{(\alpha)}\tau s_v$ in (4.7) by $z_{p,\rho}^{(\alpha)}\tau\sigma$ where $z_{p,\rho}^{(\alpha)}$ $(= t_{v,p,\rho}^{(\alpha)}$ for $v = \infty)$ denotes the upper equicoordinate α point of the p-variate equicorrelated standard normal distribution with common correlation ρ. For tables of $z_{p,\rho}^{(\alpha)}$ see Gupta, Nagel and Panchapakesan (1973). Two other relevant references are Gupta (1963) and Milton (1963).

II. Two-sided confidence intervals. When σ^2 is unknown the joint two-sided confidence intervals are given by

$$\alpha_0 - \alpha_i \in [\hat{\alpha}_0 - \hat{\alpha}_i \pm t_{v,p,\rho}^{\prime(\alpha)}\tau s_v] \quad (1 \le i \le p). \quad (4.8)$$

In (4.8) $t_{v,p,\rho}^{\prime(\alpha)}$ denotes the upper α point of the distribution of $\max\{|t_i|(1 \le i \le p)\}$ where $(t_1, \ldots, t_p)$ has a p-variate equicorrelated central t distribution with common correlation ρ, and with df v (Dunnett and Sobel 1954); for tables of $t_{v,p,\rho}^{\prime(\alpha)}$ see Hahn and Hendrickson (1971). See also, Krishnaiah, and Armitage (1970), who have tabled $(t_{v,p,\rho}^{\prime(\alpha)})^2$.

When σ^2 is known, the joint two-sided confidence intervals are obtained by replacing $t_{v,p,\rho}^{\prime(\alpha)}\tau s_v$ in (4.8) by $z_{p,\rho}^{\prime(\alpha)}\tau\sigma$ where $z_{p,\rho}^{\prime(\alpha)}$ $(= t_{v,p,\rho}^{\prime(\alpha)}$ for $v = \infty)$ denotes the upper α point of the distribution of $\max\{|Z_i|(1 \le i \le p)\}$ and $(Z_1, \ldots, Z_p)$ has a p-variate equicorrelated standard normal distribution with common correlation ρ. Hahn and Hendrickson's tables with $v = 60$ can be used here to obtain a conservative approximation to $v = \infty$.

III. Pairwise comparisons between test treatments. In Remark 4.2 we noted that the $\hat{\alpha}_i$ $(1 \le i \le p)$ have a constant variance and equal correlation when BTIB designs are used. Thus Tukey's procedure can be easily modified as described in Miller (1966, pp. 41–42) to provide $100(1-\alpha)$ percent confidence intervals for $\alpha_i - \alpha_j$ $(i \ne j,\ 1 \le i,\ j \le p)$. These are given by

$$\alpha_i - \alpha_j \in [\hat{\alpha}_i - \hat{\alpha}_j \pm q^{(\alpha)}_{\nu,p}(1-\rho)^{1/2}\tau s_\nu] \quad (i \neq j;\ 1 \le i, j \le p) \tag{4.9}$$

where $q^{(\alpha)}_{\nu,p}$ denotes the upper α point of the Studentized range distribution with parameter p and ν df. In (4.9) we made use of the fact that $\eta(1-\gamma)^{1/2} = \tau(1-\rho)^{1/2}$ from (4.5). Of course, our principal objective when designing the experiment was to estimate the $\alpha_0 - \alpha_i$ $(1 \le i \le p)$ optimally; the preceding result is a useful by-product if the experimenter is interested as well in estimating the $\alpha_i - \alpha_j$ $(i \neq j;\ 1 \le i, j \le p)$.

5. THE CLASS OF ADMISSIBLE DESIGNS

5.1 Optimal and Admissible Designs

In this section we propose a rationale for choosing a design from a set of competing BTIB designs. For simplicity of exposition we consider here the case of one-sided confidence intervals; σ^2 is assumed to be known. In Remark 5.2 we point out how our results extend to the case of two-sided confidence intervals and/or to σ^2 unknown.

We limit consideration to confidence intervals of the form $\{\alpha_0 - \alpha_i \ge \hat{\alpha}_0 - \hat{\alpha}_i - d\ (1 \le i \le p)\}$ where $d > 0$ is a specified "yardstick" associated with the common width of the confidence intervals. The probability P associated with this joint confidence statement can be written as

$$\begin{aligned} P &= \Pr\{\alpha_0 - \alpha_i \ge \hat{\alpha}_0 - \hat{\alpha}_i - d\ (1 \le i \le p)\} \\ &= \Pr\{Z_i \le d/\tau\sigma\ (1 \le i \le p)\} \\ &= \int_{-\infty}^{\infty} \Phi^p\left(\frac{x\sqrt{\rho} + d/\tau\sigma}{\sqrt{1-\rho}}\right) d\Phi(x) \end{aligned} \tag{5.1}$$

where $(Z_1, \ldots, Z_p)$ has a p-variate equicorrelated standard normal distribution with common correlation ρ, and $\Phi(\cdot)$ denotes the standard univariate normal distribution function. Note that for given p and specified d/σ the probability P of (5.1) depends on the BTIB design employed only through τ and ρ. This fact will facilitate comparisons between BTIB designs. In these comparisons we mainly restrict consideration to BTIB designs with possibly unequal b values for given (p, k); however, as pointed out in Remark 5.1, our results can be extended to the comparison of BTIB designs with unequal k values. We start by making the following definition:

Definition 5.1: For given (p, k) and specified d/σ the BTIB design that achieves a joint confidence coefficient $P \ge 1 - \alpha$ with the smallest b is said to be *optimal* for that value of $1 - \alpha$.

To determine the optimal BTIB design for given (p, k), $1 - \alpha$ and specified d/σ, one would proceed as follows: Find the design that for given (p, k, b) and d/σ maximizes P, and then vary b to find the smallest b for which the maximum P is $\ge 1 - \alpha$.

In the search for the optimal design for given (p, k), it is desirable to eliminate from consideration certain designs that are uniformly dominated by other designs and hence cannot be optimal for any d/σ or $1 - \alpha$. The definition of such an *inadmissible* design follows.

Definition 5.2: If for given (p, k) we have two BTIB designs D_1 and D_2 with parameters (b_1, τ_1^2, ρ_1) and (b_2, τ_2^2, ρ_2) with $b_1 \le b_2$, and if for *every* d and σ, D_1 yields a confidence coefficient P at least as large as (larger than) that yielded by D_2 when $b_1 < b_2$ $(b_1 = b_2)$, then we say that D_2 is *inadmissible* wrt D_1. If a design is not inadmissible, then it is *admissible.* If $b_1 = b_2$, $\tau_1^2 = \tau_2^2$, and $\rho_1 = \rho_2$, then we say that D_1 and D_2 are *equivalent.*

For given (p, k) one would limit consideration to all admissible designs in the search for optimal designs. Furthermore, if for given (p, k) we have two or more equivalent designs, then only one of such designs need be considered. It is easy to verify that a necessary and sufficient condition for BTIB designs D_1 and D_2 to be equivalent is that $(\lambda_0^{(1)}, \lambda_1^{(1)}, b_1) = (\lambda_0^{(2)}, \lambda_1^{(2)}, b_2)$ where $(\lambda_0^{(i)}, \lambda_1^{(i)}, b_i)$ corresponds to D_i $(i = 1, 2)$. For an example of equivalent designs for $(p, k, b) = (4, 3, 7)$ we have the designs of (3.2) and (3.9d), both of which have $\lambda_0 = \lambda_1 = 2$. Equivalent designs can sometimes provide flexibility to the experimenter without changing the confidence coefficient. For example, the experimenter might prefer the design of (3.9d) to the design of (3.2) if (say) the control treatments are more readily available than any of the test treatments. (See also the example of $D_1 \cup D_2$ vs. D_3 in the beginning of Section 5.2.)

The following theorem gives a characterization of inadmissibility that is easy to verify.

Theorem 5.1: For given (p, k) consider two BTIB designs D_1 and D_2 with parameters (b_1, τ_1^2, ρ_1) and (b_2, τ_2^2, ρ_2), respectively. Design D_2 is inadmissible wrt design D_1 if and only if $b_1 \le b_2$, $\tau_1^2 \le \tau_2^2$ and $\rho_1 \ge \rho_2$ with at least one inequality strict.

Proof of sufficiency: From (5.1) we see that as τ decreases for fixed d, σ and ρ (ρ increases for fixed d, σ, and τ), the confidence coefficient P increases. The monotonicity wrt ρ follows from Slepian's inequality.

Proof of necessity: Suppose that the confidence coefficient associated with D_1 is larger than the confidence coefficient associated with D_2 for every d and σ. Then $\tau_1^2 \le \tau_2^2$ $(\rho_1 \ge \rho_2)$ follows from letting $d \uparrow \infty$ $(d \downarrow 0)$.

For an application of this theorem consider the following two BTIB designs for $(p, k) = (4, 3)$:

$$D_1 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 4 \end{Bmatrix},$$

$$D_2 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 & 2 & 4 & 4 & 4 \\ 2 & 3 & 3 & 4 & 3 & 4 & 4 & 4 \end{Bmatrix}.$$

For these designs $\lambda_0^{(1)} = \lambda_0^{(2)} = 2$, $\lambda_1^{(1)} = \lambda_1^{(2)} = 2$, and $b_1 = 7 < b_2 = 8$. Hence $\tau_1^2 = \tau_2^2 = \frac{3}{5}$ and $\rho_1 = \rho_2 = \frac{1}{2}$, and thus both D_1 and D_2 yield the same P for every d and σ; however, D_2 is inadmissible wrt D_1 because D_2 requires a larger total number of observations than does D_1.

Remark 5.1: Definitions 5.1 and 5.2 can be extended to permit comparison of BTIB designs having unequal k values. Such comparisons would be of interest to the experimenter who is faced with the choice of block size, subject to the restriction that the common block size $k < p + 1$. In this case, for given p and specified d/σ the BTIB design that achieves a joint confidence coefficient $P \geq 1 - \alpha$ with the smallest $N = kb$ is said to be *optimal* for that value of $1 - \alpha$. If this more general definition of optimality is used, the characterization of inadmissibility given by Theorem 5.1 would be modified as follows: For given p consider two BTIB designs D_1 and D_2 with parameters $(b_1, k_1, \tau_1^2, \rho_1)$ and $(b_2, k_2, \tau_2^2, \rho_2)$, respectively. Design D_2 is inadmissible wrt D_1 if and only if $N_1 = k_1 b_1 \leq N_2 = k_2 b_2$, $\tau_1^2 \leq \tau_2^2$, and $\rho_1 \geq \rho_2$ with at least one inequality strict.

For an interesting application of this more general definition see Remark 2.3 in Bechhofer and Tamhane (1980a). It was shown there that if one considers designs that are admissible for $(p, k) = (4, 3)$ and designs that are admissible for $(p, k) = (4, 4)$, and then makes comparisons between the two sets of admissible designs, in general only designs with $k = 3$ can be inadmissible wrt designs with $k = 4$, and not vice versa. In fact, by means of a computer search for $(p, k) = (4, 3)$ and $(p, k) = (4, 4)$, we were able to find only one counterexample, namely,

$$D_1 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 2 & 2 & 3 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 & 3 & 4 & 4 & 4 \end{Bmatrix},$$

$$D_2 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 \end{Bmatrix}.$$

Both D_1 and D_2 have $\tau^2 = \frac{4}{10}$, $\rho = \frac{1}{2}$ and therefore achieve the same P. However, D_2 for $k = 4$ has $N = 32$, while D_1 for $k = 3$ has $N = 30$. Thus although each design is admissible for its own k value, D_2 is inadmissible wrt D_1.

Remark 5.2: The inadmissibility characterization given in Theorem 5.1 applies equally well to the case of joint two-sided confidence intervals; then the monotonicity wrt ρ follows from Šidak's (1968) results. (Of course, the optimal designs might be different in the one-sided and two-sided cases for the same (p, k) and d/σ.) The same general ideas carry over for *unknown* σ^2, except that then one would have to specify the *expected* common "width" of the confidence intervals.

5.2 Strongly (S-) Inadmissible and Combination (C-) Inadmissible Designs

The candidates for an optimal design for given (p, k) will be all admissible BTIB designs that can be constructed by forming unions of replications of all known generator designs for that given (p, k). In this section we give three rules that can be used to reduce further the number of generator designs that must be used for given (p, k) and arbitrary b to construct all admissible BTIB designs; this set of generator designs is called the *minimal complete class of generator designs*, and is formally characterized in Definition 5.6.

If the union of two or more generator designs yields an equivalent generator design, then we choose to eliminate the latter design from consideration and thereby maintain more flexibility for our construction of designs involving larger numbers of blocks. Thus, for example, for $p = 4$, $k = 4$ the designs

$$D_1 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 \end{Bmatrix}, \quad D_2 = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{Bmatrix},$$

$$D_3 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 4 \end{Bmatrix}$$

are all generator designs, but D_3 is equivalent to $D_1 \cup D_2$; hence we choose to retain only D_1 and D_2 but not D_3. This leads us to the following definition.

Definition 5.3: Suppose that for given (p, k) we have $n \geq 2$ BTIB generator designs D_i $(1 \leq i \leq n)$, no two of which are equivalent, and no one of which is equivalent to the union of replications of one or more of the other generator designs. Then $\{D_i\,(1 \leq i \leq n)\}$ is referred to as the set of *nonequivalent* generator designs.

It would be tempting to eliminate any inadmissible generator designs from the set of nonequivalent generator designs. However, it is not in general true for given (p, k) that if design D is inadmissible, then every design $D \cup D'$ is also inadmissible. Thus, for example, for $p = 4$, $k = 4$ the BTIB designs

$$D_1 = \left\{ \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 3 & 4 & 4 & 4 \end{array} \right\},$$

$$D_2 = \left\{ \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \end{array} \right\},$$

which are unions of generator designs, have $\lambda_0^{(1)} = 2$, $\lambda_1^{(1)} = 2$, $\lambda_0^{(2)} = 4$, $\lambda_1^{(2)} = 0$, and $b_1 = b_2$, and it is easy to verify that D_2 is inadmissible wrt D_1. However, $D_2 \cup D_3$ is admissible wrt $D_1 \cup D_3$ where

$$D_3 = \left\{ \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{array} \right\}.$$

Hence in this case it would not be desirable to eliminate D_2 from our set of admissible designs.

In some cases it is possible to identify those generator designs that *always* yield inadmissible or equivalent designs when unions of them are taken with other designs. This can be done using the concept of strong (S-) inadmissibility and combination (C-) inadmissibility, which are defined as follows.

Definition 5.4: If for given (p, k) we have two BTIB designs D_1 and D_2 (not necessarily generator designs), we say that D_2 is *S-inadmissible* wrt D_1 if D_2 is inadmissible wrt D_1, and if for any arbitrary BTIB design D_3 we have that $D_2 \cup D_3$ is inadmissible wrt $D_1 \cup D_3$.

Thus S-inadmissibility implies inadmissibility but not vice versa. An easily verifiable sufficient condition for S-inadmissibility of a BTIB design is given in the following theorem.

Theorem 5.2: A sufficient condition for a BTIB design D_2 to be S-inadmissible wrt a BTIB design D_1 with the same (p, k) is that $b_1 \le b_2$, $\lambda_0^{(1)} = \lambda_0^{(2)}$, $\lambda_1^{(1)} \ge \lambda_1^{(2)}$ with at least one inequality being strict; here $(\lambda_0^{(i)}, \lambda_1^{(i)}, b_i)$ is associated with D_i $(i = 1, 2)$.

Proof: If $\lambda_1^{(1)} = \lambda_1^{(2)}$ and $b_1 < b_2$, then the result is obvious. If $\lambda_1^{(1)} > \lambda_1^{(2)}$, then the result follows from the fact that for fixed λ_0 the parameter τ^2 of (4.3) is a decreasing function of λ_1, and ρ of (4.4) is an increasing function of λ_1.

As an illustration of the use of Theorem 5.1 we note that for $p = 3$, $k = 3$ the designs D_0 and D_1 of (3.5) have $\lambda_0^{(0)} = 2$. $\lambda_1^{(0)} = 1$, $b_0 = 3$, and $\lambda_0^{(1)} = 2$, $\lambda_1^{(1)} = 0$, $b_1 = 3$, respectively. Hence, D_1 is S-inadmissible wrt D_0.

There are certain BTIB designs that are not S-inadmissible as defined before but that can be deleted without any loss from our set of generator designs. The identification of such designs requires the concept of combination (C-) inadmissibility, which is more general than S-inadmissibility.

Definition 5.5: Suppose that for given (p, k) we have $n \ge 2$ generator designs D_i $(1 \le i \le n)$, which are nonequivalent, and none of which is S-inadmissible. The designs D, D', D'' described later are constructed from the designs in the set $\{D_i\ (1 \le i \le n)\}$. Consider a BTIB design D, and an *arbitrary* BTIB design D'. If for every D' there exists a BTIB design D'' such that $D \cup D'$ is either inadmissible wrt D'' or equivalent to D'', and D is not included in D'', then we say that D is C-inadmissible wrt the set $\{D_i\ (1 \le i \le n)\}$.

Remark 5.3: If a design D_i is a member of a set $\{D_1, \ldots, D_n\}$ that contains only generator designs that are nonequivalent and none of which is S-inadmissible, and if D_i is C-inadmissible wrt that set, then D_i can be deleted from the set, and we shall say that D_i is C-inadmissible wrt the set $\{D_j\ (j \ne i, 1 \le i \le n)\}$.

Remark 5.4: We point out some critical distinctions between S-inadmissible and C-inadmissible designs. First, we note that Theorem 5.2 provides an easy way of checking whether certain BTIB designs are S-inadmissible wrt certain other BTIB designs for that (p, k). On the other hand, in order to identify a C-inadmissible design it is necessary to examine every different elementary combination of generator designs, and in some cases higher order combinations, and show that each such combination leads to inadmissible or equivalent designs. Examples of C-inadmissible designs and proofs of their C-inadmissibility are given in Bechhofer and Tamhane (1979b, 1980a,b), and Bechhofer, Tamhane, and Mykytyn (1980).

We also note that unions of certain designs with C-inadmissible designs may be admissible, but each such admissible design is equivalent to some other design not involving that C-inadmissible design. Such a possibility cannot arise with an S-inadmissible design.

Finally we point out that if a design is identified as being S-inadmissible using the sufficient condition of Theorem 5.2, then that design can be permanently deleted without loss, even if it is not known whether the set $\{D_i\ (1 \le i \le n)\}$ contains *all* generator designs for given (p, k). This is in contrast to the situation concerning a C-inadmissible design, which is defined wrt the set $\{D_i\ (1 \le i \le n)\}$. A design can be C-inadmissible wrt $\{D_i\ (1 \le i \le n)\}$, but not so wrt $\{D_i\ (1 \le i \le n + 1)\}$ where this new set contains the n original designs plus one additional one and consists of $n + 1$ designs that are nonequivalent, none of which is S-inadmissible. Thus a C-inadmissible design cannot be eliminated unless it is known that $\{D_i \; (1 \le i \le n)\}$ contains *all* nonequivalent and non-S-inadmissible generator designs for the particular (p, k) of interest. For $p \ge 2$, $k = 2$ and $p = 3$, $k = 3$ these sets are given by (3.4) and D_0, D_2 of (3.5), respectively. For $(p, k) = (4, 3)$ and $(5, 3)$, and $(p, k) = (4, 4)$ this problem is considered in Bechhofer and Tamhane (1979b) and (1980a), respectively; for these cases we do not know whether we have enumerated *all* the required generator designs in each set, but we conjecture that we have done so.

We are now led to our final definition.

Definition 5.6: If the set $\{D_1, \ldots, D_n\}$ contains all generator designs for given (p, k), and if $\{D_{i_1}, \ldots, D_{i_m}\}$ with $m \le n$ is the subset that contains all nonequivalent, non-S-inadmissible and non-C-inadmissible generator designs, then the latter set will be referred to as a *minimal complete class* of generator designs for given (p, k). (This class will not be unique if one or more of the generator designs in the class can be replaced by an equivalent generator design; if this happens, the resulting class will serve equally well in the search for an optimal design.) The designs in the minimal complete class will serve as building blocks for all BTIB designs which will be of interest to us in our search for the optimal design.

We illustrate Definition 5.6 by giving in Table 2 our *conjectured* minimal complete class of generator designs for $p = 4$, $k = 3$. We have prepared corresponding tables for $(p, k) = (5, 3)$, $(6, 3)$, $(4, 4)$, $(5, 4)$, and $(6, 4)$, and using these tables we have computed catalogs of admissible designs for each b and also optimal designs for selected d/σ and $1 - \alpha$. These will appear elsewhere.

Note added in proof: When this paper was in galley proof the authors received a personal communication from William I. Notz of the Department of Statistics, Purdue University. Notz has *proved analytically* that the five generator designs in Table 2 do *not* constitute a minimal complete class for $p = 4$, $k = 3$, but that these five along with a sixth generator design that he has constructed (with parameters $b = 8$, $\lambda_0 = 1$, $\lambda_1 = 3$) *do* constitute a minimal complete class; he has also *proved* that the conjectured minimal complete class of generator designs for $p = 6$, $k = 3$ (given in Bechhofer and Tamhane 1980b) is indeed a minimal complete class. Proofs of these results and other related ones will be given elsewhere.

5.3 Relationship to Other Optimality Criteria

Kiefer (along with many others) has studied extensively the problem of optimal design in a series of papers starting with Kiefer (1958). The three main criteria considered by Kiefer are D-, A-, and E-optimality, which correspond, respectively, to minimizing

Table 2. *Conjectured Minimal Complete Class of Generator Designs for $p = 4$, $k = 3$*

Label	Design	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	$\left\{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix}\right\}, \left\{\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{matrix}\right\}$	4	2	0
D_2	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 \end{matrix}\right\}$	6	3	1
D_3	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 4 \end{matrix}\right\}, \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & 4 \\ 3 & 4 & 3 & 4 & 3 & 4 & 4 \end{matrix}\right\}$	7	2	2
D_4	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 3 & 3 \\ 2 & 2 & 3 & 4 & 3 & 4 & 3 & 4 & 4 & 4 \end{matrix}\right\}$	10	4	2
D_5	$\left\{\begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{matrix}\right\}$	4	0	2

the determinant, trace, and the maximum eigen value of the variance-covariance matrix of the BLUE of the parameter vector of interest which in our case is $(\alpha_0 - \alpha_1, \ldots, \alpha_0 - \alpha_p)'$. The eigen values (neglecting constant proportionality factors) of the variance-covariance matrix of $(\hat{\alpha}_0 - \hat{\alpha}_1, \ldots, \hat{\alpha}_0 - \hat{\alpha}_p)'$ can be obtained from (4.2) through (4.4); they are $(\lambda_0 + p\lambda_1)^{-1}$ of multiplicity $p - 1$, and λ_0^{-1} of multiplicity 1. Thus for given (p, k, b) the three criteria can be stated as (a) D-optimality: minimize $\{\lambda_0(\lambda_0 + p\lambda_1)^{p-1}\}^{-1}$, that is, maximize

$$\{\lambda_0(\lambda_0 + p\lambda_1)^{p-1}\};$$

(b) A-optimality: minimize

$$\{\lambda_0^{-1} + (p - 1)(\lambda_0 + p\lambda_1)^{-1}\};$$

(c) E-optimality: minimize λ_0^{-1}, that is, maximize λ_0.

Although these criteria are simpler than ours, it should be kept in mind that they refer to an *ellipsoidal* joint confidence region for $(\alpha_0 - \alpha_1, \ldots, \alpha_0 - \alpha_p)'$. In the case of MCC such a confidence region is of less interest than the "rectangular" confidence regions that we have proposed, and are commonly used. Hence we do not consider the Kiefer criteria further here.

6. CONCLUDING REMARKS AND DIRECTIONS OF FUTURE RESEARCH

In this paper we introduced a new general class of incomplete block designs that are appropriate for use in the MCC problem. We refer to these as balanced treatment incomplete block (BTIB) designs. The basic results concerning the structure of such designs are derived, and the properties of the relevant estimates obtained with such designs are given. Admissibility and inadmissibility of these designs are defined, and these criteria are used to eliminate inferior designs. In the search for optimal designs it suffices to restrict consideration to admissible designs. It is shown how the concepts of S-inadmissibility and C-inadmissibility can be used to obtain a minimal complete class of generator designs from which catalogs of admissible designs can be constructed.

The combinatorial problem of constructing *all* BTIB designs for given (p, k, b), and the procedure for choosing an *optimal* design from such a set, are not solved in the present paper. However, some methods of design construction are given. The aforementioned problems are related in the sense that to solve the optimization part completely one must have constructed most, if not all, generator designs for given (p, k); the problem of determining how many generator designs exist for arbitrary (p, k), and then enumerating them, appears to be a very formidable one. Alternatively, the problem of construction of BTIB designs for arbitrary (p, k, b) can be set up in the manner of Foody and Hedayat (1977, Lemma 4.1), which is suitable for a solution on a computer. However, there is no guarantee that all BTIB designs can be generated in this way. Also, the magnitude of difficulty of our problem is substantially greater than theirs, and therefore a solution by way of this route looks rather remote at this stage.

For $p \geq 2$, $k = 2$ and $p = 3$, $k = 3$ the situation is very simple since it is necessary to consider essentially only two generator designs for each (p, k) case; these cases are considered in detail in Bechhofer and Tamhane (1979a) and the optimal design is given for a large range of the useful (p, k, b)- and d/σ-values. For $p = 4, 5, 6$ and $k = 3, 4$ it is possible to enumerate most of the generator designs; these cases are considered in Bechhofer and Tamhane (1979b, 1980a, 1980b), Bechhofer, Tamhane, and Mykytyn (1980) along with the associated optimal designs.

7. ACKNOWLEDGMENTS

The writers would like to thank Jack Kiefer for suggesting the necessity part of Theorem 5.1, and Carl Emont and Stephen Mykytyn for the fine computer programs that they produced, and also for their insights, which led to our definition of BTIB designs and C-inadmissible designs, respectively. We also wish to acknowledge the contribution of a referee whose helpful comments enhanced the practical value of our results.

This research was supported by U.S. Army Research Office-Durham contract DAAG29-80-C-0036 and Office of Naval Research Contract N00014-75-C-0586 at Cornell University, and NSF Grant ENG77-06112 at Northwestern University.

8. APPENDIX

Proof of Theorem 3.1. For given (p, k, b) consider an arbitrary design with the incidence matrix $\{r_{ij}\}$. Then it is well known (see Eq. (3.1) of Kiefer 1958) that the information matrix $\mathbf{M} = \{m_{i_1 i_2}\}$ of $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_p)'$ is given by

$$m_{i_1 i_2} = \begin{cases} r_{i_1} - \dfrac{1}{k} \displaystyle\sum_{j=1}^{b} r_{i_1 j}^2 & (i_1 = i_2) \\ -\dfrac{\lambda_{i_1 i_2}}{k} & (i_1 \neq i_2). \end{cases} \tag{A.1}$$

Note that $\mathbf{M}$: $(p + 1) \times (p + 1)$ is a singular matrix and $\sum_{i_2=0}^{p} m_{i_1 i_2} = 0$ for each i_1. We require the information matrix of $\mathbf{U}\boldsymbol{\alpha} = (\alpha_0 - \alpha_1, \ldots, \alpha_0 - \alpha_p)'$ where $\mathbf{U} = \{u_{i_1 i_2}\}$: $p \times (p + 1)$ is given by

$$u_{i_1 i_2} = \begin{cases} 1 & (i_2 = 1, i_1 = 1, \ldots, p) \\ -1 & (i_2 = i_1 + 1, i_1 = 1, \ldots, p) \\ 0 & \text{otherwise.} \end{cases}$$

To avoid computing the generalized inverse of $\mathbf{M}$, we shall employ the following method given to obtain the desired information matrix. Let $\mathbf{Q}$: $p \times (p + 1)$ be any matrix the rows of which form p orthogonal contrasts. Then we can write $\mathbf{U}$ as $\mathbf{U} = \mathbf{PQ}$ where $\mathbf{P}$: $p \times p$ is a nonsingular matrix. The information matrix of $\mathbf{Q}\boldsymbol{\alpha}$ is given by $\mathbf{QMQ}'$. Therefore, the variance-covariance matrix (except for the common factor σ^2) of $\mathbf{U}\hat{\boldsymbol{\alpha}}$ is given by $\mathbf{P}(\mathbf{QMQ}')^{-1}\mathbf{P}'$. Hence the information matrix $\mathbf{M}^*$ of $\mathbf{U}\boldsymbol{\alpha}$ is given by

$$\begin{aligned} \mathbf{M}^* &= [\mathbf{P}(\mathbf{QMQ}')^{-1}\mathbf{P}']^{-1} \\ &= (\mathbf{P}')^{-1}(\mathbf{QMQ}')\mathbf{P}^{-1} \\ &= \mathbf{V}'\mathbf{M}\mathbf{V} \end{aligned} \tag{A.2}$$

where $\mathbf{V} = \mathbf{Q}'\mathbf{P}^{-1}$ satisfies $\mathbf{UV} = \mathbf{I}$, with $\mathbf{I}$ being the $p \times p$ identity matrix. It can be easily verified that $\mathbf{V} = \{v_{i_1 i_2}\}$: $(p + 1) \times p$ is given by

$$v_{i_1 i_2} = \begin{cases} 0 & (i_1 = i_2 + 1, i_2 = 1, \ldots, p), \\ 1 & \text{otherwise.} \end{cases}$$

Substituting for $\mathbf{V}$ in (A.2), the information matrix $\mathbf{M}^* = \{m_{i_1 i_2}^*\}$ of $\mathbf{U}\boldsymbol{\alpha}$ can be written as

$$\begin{aligned} m_{i_1 i_2}^* &= \sum_{\substack{g=0 \\ g \neq i_1}}^{p} \sum_{\substack{h=0 \\ h \neq i_2}}^{p} m_{gh} \\ &= m_{i_1 i_2} \quad (i_1, i_2 = 1, \ldots, p) \end{aligned} \tag{A.3}$$

where (A.3) follows from the fact that the rows and columns of $\mathbf{M}$ sum to zero. For a design to be BTIB, the matrix $\mathbf{M}^*$ must be completely symmetric (i.e., all diagonal elements of $\mathbf{M}^*$ must be equal, and all off-diagonal elements must be equal). Therefore, we have

$$m_{11} = m_{22} = \cdots = m_{pp}$$

and (A.4)

$$m_{12} = m_{13} = \cdots = m_{p-1,p}.$$

Using expression (A.1) for $m_{i_1 i_2}$ in (A.4) implies (3.1). Denoting the common value of the λ_{0i} ($1 \le i \le p$) by λ_0, and the common value of the $\lambda_{i_1 i_2}$ ($i_1 \ne i_2$; $1 \le i_1$, $i_2 \le p$) by λ_1, we see that

$$m_{i_1 i_2} = -\lambda_1/k \quad (i_1 \ne i_2;\ 1 \le i_1, i_2 \le p)$$

and

$$m_{ii} = \{\lambda_0 + (p-1)\lambda_1\}/k \quad (1 \le i \le p).$$

Thus

$$\mathbf{M}^* = \{(\lambda_0 + p\lambda_1)\mathbf{I} - \lambda_1 \mathbf{J}\}/k \tag{A.5}$$

where $\mathbf{J}$: $p \times p$ is a matrix consisting only of 1's. The inverse of $\mathbf{M}^*$ is the desired variance-covariance matrix of $\mathbf{U}\hat{\boldsymbol{\alpha}}$; the expressions for (4.2) and (4.4) are then easily obtained from $(\mathbf{M}^*)^{-1}$.

Derivation of the expressions for the estimates. The normal equations for the least squares estimates (BLUE) $\hat{\mu}$, $\hat{\alpha}_i$, $\hat{\beta}_j$ of μ, α_i, β_j, respectively, ($0 \le i \le p$, $1 \le j \le b$) are

$$N\hat{\mu} + \sum_{i=0}^{p} r_i\hat{\alpha}_i + k\sum_{j=1}^{b}\hat{\beta}_j = G \tag{A.6}$$

$$r_i\hat{\mu} + r_i\hat{\alpha}_i + \sum_{j=1}^{b} r_{ij}\hat{\beta}_j = T_i \qquad (0 \le i \le p) \tag{A.7}$$

$$k\hat{\mu} + \sum_{i=0}^{p} r_{ij}\hat{\alpha}_i + k\hat{\beta}_j = B_j \qquad (1 \le j \le b) \tag{A.8}$$

where T_i and B_j are defined in Section 4.1, and $G = \sum_{i=0}^{p} T_i = \sum_{j=1}^{b} B_j$. Substituting

$$\hat{\mu} + \hat{\beta}_j = \left(B_j - \sum_{i=0}^{p} r_{ij}\hat{\alpha}_i\right)\Big/k \tag{A.9}$$

from (A.8) into (A.7), we obtain

$$r_i\hat{\alpha}_i + \frac{1}{k}\sum_{j=1}^{b} r_{ij}B_j - \frac{1}{k}\sum_{h=0}^{p}\hat{\alpha}_h\sum_{j=1}^{b} r_{hj}r_{ij} = T_i. \tag{A.10}$$

Now $\sum_{j=1}^{b} r_{ij}B_j = B_i^*$ and for $h \ne i$ we have that $\sum_{j=1}^{b} r_{hj}r_{ij} = \lambda_0(\lambda_1)$ if only *one* of h and $i = 0$ (if *both* h and $i \ne 0$). From (A.10) for $i = 0$ we obtain

$$r_0\hat{\alpha}_0 + \frac{B_0^*}{k} - \frac{1}{k}\left\{\hat{\alpha}_0\sum_{j=1}^{b} r_{0j}^2 + \lambda_0\sum_{h=1}^{p}\hat{\alpha}_h\right\} = T_0. \tag{A.11}$$

Using the fact that $\sum_{h=1}^{p}\hat{\alpha}_h = -\hat{\alpha}_0$ and that $r_0 - 1/k \ge \sum_{j=1}^{b} r_{0j}^2 = m_{00} = p\lambda_0/k$, from (A.11) we obtain

$$\hat{\alpha}_0 = \frac{Q_0}{(p+1)\lambda_0}. \tag{A.12a}$$

In the same way we obtain

$$\hat{\alpha}_i = \frac{Q_i + (\lambda_0 - \lambda_1)\hat{\alpha}_0}{(\lambda_0 + p\lambda_1)} \quad (1 \le i \le p). \tag{A.12b}$$

Combining (A.12a) and (A.12b), we obtain (4.1).

Derivation of the formula for the adjusted treatment sum of squares in the analysis of variance table (Table 1). Following the Scheffé (1959) notation, let $\mathscr{S}_\Omega$ denote the minimum error sum of squares (SS) under the assumptions Ω of Section 2, and let $\mathscr{S}_\omega$ denote the minimum error SS under $\omega = H \cap \Omega$ where the hypothesis is H: $\alpha_i = 0$ ($0 \le i \le p$). Then the SS of treatments adjusted for blocks is given by

$$SS_{\text{treat(adj.)}} = \mathscr{S}_\omega - \mathscr{S}_\Omega. \tag{A.13}$$

Now

$$\begin{aligned}\mathscr{S}_\Omega &= SS_{\text{error}} \\ &= \sum_{i=0}^{p}\sum_{j=1}^{b}\sum_{h=1}^{k}(y_{ijh} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j)^2 I_{ijh}\end{aligned} \tag{A.14}$$

where $I_{ijh} = 1$ if the ith treatment is assigned to the hth plot of the jth block ($0 \le i \le p$, $1 \le j \le b$, $1 \le h \le k$) and $= 0$ otherwise. Substituting $\hat{\beta}_j$ from (A.9) into (A.14) and expanding and simplifying the resulting square, we obtain

$$\begin{aligned}\mathscr{S}_\Omega = H &- \frac{1}{k}\sum_{j=1}^{b} B_j^2 + \sum_{i=0}^{p} r_i\hat{\alpha}_i^2 \\ &- \frac{1}{k}\sum_{j=1}^{b}\left(\sum_{i=0}^{p} r_{ij}\hat{\alpha}_i\right)^2 - Q\end{aligned} \tag{A.15}$$

where $H = \sum_{i=0}^{p}\sum_{j=1}^{b}\sum_{h=1}^{k} y_{ijh}^2 I_{ijh}$ and

$$Q = \frac{2}{k}\sum_{i=0}^{p}\hat{\alpha}_i Q_i.$$

Under ω, we have $\hat{\mu} = G/N$ and $\hat{\beta}_j = B_j/k - \hat{\mu}$; hence

$$\mathscr{S}_\omega = H - \frac{1}{k}\sum_{j=1}^{b} B_j^2. \tag{A.16}$$

Subtracting (A.15) from (A.16), we have from (A.13) that

$$
\begin{aligned}
SS_{\text{treat(adj.)}} &= Q + \frac{1}{k}\sum_{j=1}^{b}\left(\sum_{i=0}^{p} r_{ij}\hat{\alpha}_i\right)^2 \\
&\quad - \sum_{i=0}^{p} r_i \hat{\alpha}_i^2 \\
&= Q - \sum_{i=0}^{p}\left(r_i - \frac{1}{k}\sum_{j=1}^{b} r_{ij}^2\right)\hat{\alpha}_i^2 \\
&\quad + \frac{2}{k}\sum_{i_1=0}^{p}\sum_{i_2=i_1+1}^{p}\hat{\alpha}_{i_1}\hat{\alpha}_{i_2}\lambda_{i_1 i_2} \\
&= Q - \frac{p\lambda_0}{k}\hat{\alpha}_0^2 \\
&\quad - \frac{\{\lambda_0 + (p-1)\lambda_1\}}{k}\sum_{i=1}^{p}\hat{\alpha}_i^2 \\
&\quad + \frac{2\lambda_0\hat{\alpha}_0}{k}\sum_{i=1}^{p}\hat{\alpha}_i \\
&\quad + \frac{2\lambda_1}{k}\sum_{i_1=1}^{p}\sum_{i_2=i_1+1}^{p}\hat{\alpha}_{i_1}\hat{\alpha}_{i_2} \\
&= Q - \frac{(p+2)\lambda_0\hat{\alpha}_0^2}{k} \\
&\quad - \frac{(\lambda_0 + p\lambda_1)}{k}\sum_{i=1}^{p}\hat{\alpha}_i^2 \\
&\quad + \frac{\lambda_1}{k}\left(\sum_{i=1}^{p}\hat{\alpha}_i\right)^2 \\
&= Q - \frac{[(p+2)\lambda_0 - \lambda_1]\hat{\alpha}_0^2}{k} \\
&\quad - \frac{(\lambda_0 + p\lambda_1)}{k}\sum_{i=1}^{p}\hat{\alpha}_i^2. \qquad \text{(A.17)}
\end{aligned}
$$

In the preceding equations we have used the relations $r_i - 1/k \sum_{j=1}^{b} r_{ij}^2 = m_{ii}$ (from (A.1)) where $m_{00} = p\lambda_0/k$, $m_{ii} = \{\lambda_0 + (p-1)\lambda_1\}/k$ $(1 \le i \le p)$, and $\sum_{i=1}^{p} \hat{\alpha}_i = -\hat{\alpha}_0$. Substituting in (A.17) for $\hat{\alpha}_0$ from (A.12a) and for $\hat{\alpha}_i$ $(1 \le i \le p)$ from (A.12b) and after some tedious algebra, we obtain the expression for $SS_{\text{treat(adj.)}}$ as given in Table 1. The other SS expressions and the df in the table are obtained in a straightforward way.

[Received April 1979. Revised April 1980.]

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